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Vector summation within minimal angle

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Abstract

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It is shown that any finite family of vectors in the plane with sum zero can be summed in such an order that all partial sums are inside an angle of size at most $\pi/3$ with vertex at the origin, and the size $\pi/3$ is the minimum. Such a family in three-dimensional space can be summed inside on octant.

1. Introduction

Given an m-dimensional space R^m and a norm s in it, a family of vectors $V=\{v^1,\ldots,v^n\}\subset R^m$ such that $\sum v^j=0$ and $\|v^j\|_s\leq 1$, $\forall j$, is called an s-family. A family V with property $\sum v^j=0$ is called a Z-family. For a given Z-family $V=\{v^1,\ldots,v^n\}$ and an arbitrary permutation $p=(p(1),\ldots,p(n))$ of $\{1,2,\ldots,n\}=N$ compute the partial sums $w_k^p=\sum_{j=1}^k v^{p_j}$, $k\in N$. The closed polygon connecting the points 0, $w_1^p, w_2^p,\ldots,w_n^p=0$ is called a summing trajectory of the Z-family according to the permutation p. Thus, the kth edge of the trajectory is the vector v^{p_k}.

In 1913 Steinitz [19] proved that for any s-family of vectors in R^m there exists a summing trajectory which is inside a ball of a limited radius. (The latter is independent of the number of vectors and depends only on the norm and the dimension of the space.) Further efforts of many mathematicians ([4, 6–10, 12, 16]) were directed to search of the minimum radius of a ball including a

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summing trajectory of any s-family in R^m. In [1] the desired minimum radii have been found for some norms in R^2. Then, needs of functional analysis gave life to another variant of the problem where a summing trajectory was restricted by each coordinate independently. (Thus, one needs to include a trajectory inside a given parallelotope.) For example, the absolute value of the first coordinate must be bounded by the unit while each of the rest ones–by a constant. Summing the vectors of any l_∞-family in R^m in this way was shown in [3, 15, 17], while the minimal parallelotope in R^2 was found in [2].

The connection between some scheduling problems and the vector summation problem in its initial formulation (to minimize the radius of the ball) was discovered in 1974 [5, 11]. It permits one to construct good approximation schedules in polynomial time. The connection was afterwards extended to many other scheduling problems of the flow shop-, job shop- and open shop-type [14]. At the same time, it became clear that the best accuracy bounds could be obtained by means of a vector summation not within a ball in R^m but in some other regions of the space, individual for each problem (such as a right triangle or a hexagon in the plane [18]). In certain cases (e.g. in case of three-machine flow shop) the best possible bound was obtained by means of a vector summation in a given corner of the plane [13]. From this, the following interesting question arises. What is the minimum size of the angle if the disposition of the angle is not fixed in advance? It is shown in the present paper that it can always be done within the angle $\pi/3$, and the size $\pi/3$ is minimum. It is proved also that any Z-family in R^3 can be summed within an octant.

As a by-product, it is proved that for any Z-family in the plane there exists a summing trajectory whose convex hull is a triangle. It is unknown to the authors whether the number of vertices of the convex hull of a summing trajectory can be bounded from above by any constant in the three-dimensional case.

2. Vector summation in the plane

Let a Z-family $V = \{v^1, \ldots, v^n\}$ be given in the plane. An angle $\alpha = \alpha(V)$ is called a *summing angle* of the family V if there exists a permutation p of V for which the summing trajectory is wholly inside an angle of size α with vertex at the origin, and for smaller angles this does not hold.

Theorem 2.1. *For any Z-family $V \subset R^2$,*

$$\alpha(V) \leqslant \pi/3, \tag{1}$$

and the bound is best possible.

Some simple statements precede the proof of the theorem.

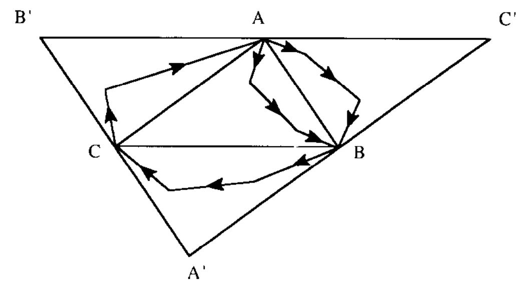

Fig. 1.

Statement 2.2. *Let a Z-family in R^2 and its summing trajectory be given. The replacement of any piece of the trajectory by its inverse is the reflection of the piece relative to the middle point of the segment connecting its end points. In particular, the replacement of the permutation by its inverse yields the trajectory, which is centrally-symmetric to the initial one.*

Lemma 2.3. *For any Z-family $V \subset R^2$ there exists a summing trajectory whose convex hull is a triangle.*

Proof. Take an arbitrary Z-family of vectors $V \subset R^2$ and order them by their directions clockwise starting with an arbitrary vector. The corresponding summing trajectory forms a convex polygon M. Inscribe into M the triangle T of the maximum area. This can be made in such a way that the vertices A, B, C of the triangle T coincide with some vertices of M. Draw the lines through each vertex of T parallel to the opposite side. These straight lines form the triangle $A'B'C'$ similar to ABC and containing the polygon M (in view of T's area being maximum), see Fig. 1.

Consider the piece of the trajectory connecting points A and B. It is wholly inside the triangle ABC' similar to ABC. Replace the summing order of this part of the trajectory by its inverse. In accordance with Statement 2.2, the new version of this part is inside the triangle ABC (being reflected relative to the center of the segment AB). Doing the same thing for the remaining two pieces of the trajectory we obtain a trajectory whose convex hull is the triangle ABC. □

Lemma 2.3 and Statement 2.2 imply the following.

Corollary 2.4. *For any Z-family $V \subset R^2$, there exist three lines passing through the origin such that V can be summed inside any one of the six angles that the lines determine.*

Proof of Theorem 2.1. The bound (1) immediately follows from corollary 2.4. It is attained in the case of three vectors of equal length with the angle $2\pi/3$ between each pair of the vectors. □

3. Vector summation inside an octant

A *quadrant* is an arbitrary right angle with vertex at the origin. A trihedral angle formed by three quadrants is called an *octant.*

Theorem 3.1. *For any Z-family of vectors $V \subset R^3$ there exists a summing trajectory which is included in an octant.*

The proof follows.

Lemma 3.2. *Let three red lines and three blue lines pass through the origin in the plane. There exists a quadrant containing two red rays and two blue rays.*

Proof of Lemma 3.2. Among the six angles that the three red lines determine, there are two unobtuse ones with a common side. Let this side be a ray of a red line X, and let Y be a line perpendicular to X and passing through the origin. These two lines, X and Y, determine four quadrants, each containing two red rays (remark that a quadrant contains both of its sides). Evidently, one of the quadrants contains two blue rays as well. □

Proof of Theorem 3.1. Choose a subset of indices $I = \{i_1, \ldots, i_k\} \subset N$ for which the partial sum $v_I = \sum_{j \in I} v^j$ has the maximum length, and denote $V_I = \{v^j, j \in I\}$. It is clear that for $J = N \setminus I = \{j_1, \ldots, j_{n-k}\}$ the equality $v_J = -v_I$ holds. Furthermore, the (orthogonal) projection of any vector v^j to the vector v_I is positive for $j \in I$ and negative for $j \in J$. Consider the projections $\{v_j^*, j \in N\}$ of the vectors V to the plane P perpendicular to v_I. It is clear, the families $V_I^* = \{v_j^*, j \in I\}$, $V_J^* = \{v_j^*, j \in J\}$ are Z-families in the plane P. Hence, there exist three straight lines for the family V_I^* (we call them 'blue') and three 'red' straight lines for the family V_J^*, guaranteed by Corollary 2.4. In accordance with Lemma 3.2, in the plane P, we find a quadrant γ containing some 'blue' angle α (placed between two 'blue' rays) and some 'red' angle β. According to Corollary 2.4, we find a permutation p_1 of vectors V_I which ensures summation of their projections V_I^* inside the angle α, and hence–inside the quadrant γ. Similarly, we compute the permutation p_2 with this property for the vectors V_J. The octant we need is bounded by the quadrant γ and by two planes perpendicular to P and containing the sides of the quadrant. It is clear that the summing trajectory $v^{p_1(i_1)} \ldots v^{p_1(i_k)} v^{p_2(j_1)} \ldots v^{p_2(j_{n-k})}$ does not go out of the octant. □

The following question seems to be interesting.

Problem 3.3. Does there exist a constant C such that for any Z-family $V \subset R^3$ there is a summing trajectory whose convex hull has at most C vertices?

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